

Some Integrals Involving Associated Legendre Functions

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Abstract. Calculations of some uncommon integrals involving Legendre functions and their derivatives, which may not be readily evaluated using known results, are presented. Some results show a special type of orthogonality relation in a certain sense. A few of these integrals find their applications in diffraction or scattering problems.

1. Introduction. In the analysis [1] of the scattering of a cylindrical electromagnetic wave by a spherical object, one encounters some special type of integrals involving associated Legendre functions. In the process of computing these integrals, one finds a number of related integrals as by-products, which may not be readily evaluated using the known results. Once a few of these integrals are calculated, the rest may be found by using the recurrence relations or other known results. For the sake of brevity, the proofs of only the primary results will be sketched and the rest will simply be presented.

2. Some Preliminary Results. Let us consider the following recurrence relations [2]:

$$(1) \quad \begin{aligned} (1 - x^2)^{1/2} \cdot P_n^{m-1}(x) &= [P_{n+1}^m(x) - P_{n-1}^m(x)] / (2n + 1) \\ &= [xP_n^m(x) - P_{n-1}^m(x)] / (n - m + 1), \end{aligned}$$

where $m, n = 0, 1, 2, \dots$.

The following relation can easily be established:

$$(2) \quad \begin{aligned} \int_{-1}^1 \frac{xP_n^m(x)P_{n+2l+1}^m(x) dx}{1 - x^2} - \int_{-1}^1 \frac{P_n^m(x)P_{n+2l}^m(x) dx}{1 - x^2} \\ = (n + 2l - m + 2) \int_{-1}^1 \frac{P_n^m(x)P_{n+2l+1}^{m-1}(x) dx}{(1 - x^2)^{1/2}}, \quad \text{where } l = 0, 1, 2, \dots \end{aligned}$$

None of the integrals appearing in (2) can be calculated easily using the tables of integrals. Therefore, we shall seek to replace the associated Legendre function $P_n^m(x)$ by its equivalent Rodrigues' formula defined by

$$(3) \quad P_k^m(x) = \frac{(1 - x^2)^{m/2}}{2^k \cdot k!} \cdot \frac{d^{m+k}}{dx^{m+k}} (x^2 - 1)^k, \quad k = 0, 1, 2, \dots,$$

and then perform repeated integration by parts. For this purpose, let us first consider the integral on the right-hand side of Eq. (2). This can be expressed in the following manner:

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$$(4i) \quad \int_{-1}^1 \frac{P_n^m(x) P_{n+2l+1}^{m-1}(x) dx}{(1-x^2)^{1/2}} = (-1)^{m-1} A_l / [2^{2n+2l+1} \cdot n! (n+2l+1)!],$$

where

$$(4ii) \quad A_l = \int_{-1}^1 dx \left[\frac{d^{n+2l+m}}{dx^{n+2l+m}} (x^2 - 1)^{n+2l+1} \right] \cdot \left[(x^2 - 1)^{m-1} \cdot \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \right].$$

Since, at $x = \pm 1$, the quantities

$$\frac{d^{n+m+2l-i}}{dx^{n+m+2l-i}} (x^2 - 1)^{n+2l+1} \quad \text{and} \quad \frac{d^{i-1}}{dx^{i-1}} \left\{ (x^2 - 1)^{m-1} \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \right\}$$

vanish for $m \leq i$ and $m \geq i + 1$ respectively (with $i = 0, 1, 2, \dots$), $(n + m + 2l)$ -times repeated integrations by parts, starting with the integration of the first factor in the integrand of (4ii), result in the following expression for A_l :

$$(5) \quad A_l = (-1)^{n+m+2l} \int_{-1}^1 dx (x^2 - 1)^{n+2l+1} \cdot \frac{d^{n+m+2l}}{dx^{n+m+2l}} \left[(x^2 - 1)^{m-1} \cdot \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \right].$$

By employing the Leibnitz theorem for derivatives of products, it can be shown that $A_l = 0$, for all $l = 0, 1, 2, \dots$. Thus it is proved that

$$(6) \quad \int_{-1}^1 \frac{P_n^m(x) P_{n+2l+1}^{m-1}(x) dx}{(1-x^2)^{1/2}} = 0, \quad l = 0, 1, 2, \dots$$

Then, from (2) and (6), we have the following result:

$$(7i) \quad I_l = \hat{I}_l, \quad l = 0, 1, 2, \dots,$$

where

$$(7ii) \quad I_l = \int_{-1}^1 \frac{x P_n^m(x) P_{n+2l+1}^m(x) dx}{1-x^2},$$

$$(7iii) \quad \hat{I}_l = \int_{-1}^1 \frac{P_n^m(x) P_{n+2l}^m(x) dx}{1-x^2}.$$

With the aid of Eqs. (1) and (6), the following relations can be established:

$$(8i) \quad \hat{I}_{l+1} = \hat{I}_l, \quad l = 0, 1, 2, \dots,$$

i.e.

$$(8ii) \quad \hat{I}_0 = \hat{I}_1 = \hat{I}_2 = \dots = \hat{I}_l = \dots$$

Similarly, one finds also (using (7i) and (8i))

$$(9) \quad I_0 = I_1 = I_2 = \dots = I_l = \dots$$

In view of the following recurrence relation

$$(10) \quad (1-x^2)^{1/2} \cdot P_k^{m+1}(x) = (k+m)P_{k-1}^m(x) - (k-m)xP_k^m(x),$$

and the equality (7i), we have

$$(11i) \quad I_l = \hat{I}_l = \hat{\hat{I}}_l / 2m, \quad m \neq 0,$$

where

$$(11\text{ii}) \quad \hat{I}_l = \int_{-1}^1 \frac{P_n^m(x)P_{n+2l+1}^{m+1}(x) dx}{(1-x^2)^{1/2}}, \quad l = 0, 1, 2, \dots$$

Let us now define the following quantities:

$$(12\text{i}) \quad J_l = \int_{-1}^1 x P_n^m(x)P_{n-2l-1}^m(x)/(1-x^2) \cdot dx,$$

$$(12\text{ii}) \quad \hat{J}_l = \int_{-1}^1 dx P_n^m(x)P_{n-2l}^m(x)/(1-x^2),$$

$$(12\text{iii}) \quad \hat{\hat{J}}^l = \int_{-1}^1 dx P_n^m(x)P_{n-2l-1}^{m-1}(x)/(1-x^2)^{1/2}.$$

Now employing the relations

$$(13\text{i}) \quad (1-x^2)^{1/2} \cdot P_k^{m+1}(x) = (k+m+1)xP_k^m(x) - (k-m+1)P_{k+1}^m(x),$$

and

$$(13\text{ii}) \quad (1-x^2)^{1/2} \cdot P_k^{m-1}(x) = [P_{k+1}^m(x) - xP_k^m(x)]/(k+m),$$

one can easily obtain the following results:

$$(14\text{i}) \quad \frac{(n+m-2l)}{(n-m-2l)} J_l - \hat{J}_l = \frac{1}{(n-m-2l)} \int_{-1}^1 \frac{P_n^m(x)P_{n-2l-1}^{m+1}(x) dx}{(1-x^2)^{1/2}},$$

and

$$(14\text{ii}) \quad \hat{J}_l - J_l = (n+m-2l-1)\hat{\hat{J}}^l, \quad (n-m-1)/2 \geq l \geq 0.$$

By appropriate choice of indices, n, m and l , it follows from (6), that

$$(15) \quad \int_{-1}^1 \frac{P_n^m(x)P_{n-2l-1}^{m+1}(x) dx}{(1-x^2)^{1/2}} = 0.$$

Then the relations (14i) and (14ii) reduce to

$$(16) \quad J_l = \frac{(n-m-2l)}{(n+m-2l)} \hat{J}_l = \frac{(n+m-2l-1)(n-m-2l)}{2m} \hat{\hat{J}}^l, \quad m \neq 0.$$

3. A Theorem. We now prove that, for integer values of n, m and s ,

$$(17\text{i}) \quad F_s = \int_{-1}^1 \frac{x}{1-x^2} \cdot P_n^m(x)P_s^m(x) dx$$

$$(17\text{ii}) \quad = \begin{cases} ((n+m)!/[m(n-m)!])\delta_{s,n+2l+1}, & \text{if } s > n, \\ ((n+m-2l-1)!/[m(n-m-2l-1)!])\delta_{s,n-2l-1}, & \text{if } s < n, \\ & l = 0, 1, 2, \dots, \text{ and } m \neq 0, \end{cases}$$

where the Kronecker delta function $\delta_{i,j}$ has its usual meaning.

Let us consider that m and $n (> m)$ are fixed and s is a variable index. If $s = n \pm 2l$, then the integrand becomes an odd function of x and consequently the integral

vanishes. However, the integral does not vanish for $s = n \pm (2l + 1)$. This establishes the validity of the introduction of the Kronecker delta function in (17i) and (17ii). In order to compute the integral for various s , let us first take $s = n + 1$ (i.e., $l = 0$) and replace the associated Legendre functions by the Rodrigues' formula (3). Then we have

$$(18i) \quad F_{n+1} = \int_{-1}^1 x P_n^m(x) P_{n+1}^m(x) / (1 - x^2) dx = \frac{(-1)^{m-1}}{2^{2n+1} \cdot n! (n + 1)!} \cdot \int_{-1}^1 x(x^2 - 1)^{m-1} \cdot \frac{d^{n+m+1}}{dx^{n+m+1}} (x^2 - 1)^{n+1} \cdot \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n dx.$$

Writing $x(x^2 - 1)^{m-1} = (1/2m)(d(x^2 - 1)^m/dx)$, $m \neq 0$, and then integrating by parts once, we have, from (18i),

$$(18ii) \quad F_{n+1} = \frac{(-1)^m}{2^{2n+2} \cdot m \cdot n! (n + 1)!} \cdot [\mathcal{G}_1 + \mathcal{G}_2],$$

where

$$(18iii) \quad \mathcal{G}_1 = \int_{-1}^1 (x^2 - 1)^m \cdot \frac{d^{n+m+2}}{dx^{n+m+2}} (x^2 - 1)^{n+1} \cdot \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n dx,$$

and

$$(18iv) \quad \mathcal{G}_2 = \int_{-1}^1 (x^2 - 1)^m \cdot \frac{d^{n+m+1}}{dx^{n+m+1}} (x^2 - 1)^{n+1} \cdot \frac{d^{n+m+1}}{dx^{n+m+1}} (x^2 - 1)^n dx.$$

For the integral \mathcal{G}_1 , integrating by parts $(n + m)$ -times starting with the integration of the factor $d^{n+m}(x^2 - 1)^n/dx^{n+m}$ and then using in the resulting expression the Leibnitz theorem for derivatives of products, we have

$$(18v) \quad \mathcal{G}_1 = (-1)^{n+m} \cdot \sum_{j=0}^{n+m} \frac{(n + m)!}{j! (n + m - j)!} \cdot \int_{-1}^1 dx (x^2 - 1)^n \left\{ \frac{d^{n+m-i}}{dx^{n+m-i}} (x^2 - 1)^m \right\} \left\{ \frac{d^{n+m+2+i}}{dx^{n+m+2+i}} (x^2 - 1)^{n+1} \right\}.$$

A little effort will show that all the terms of the above series in (18v) vanish except for $j = n - m$. Thus the contribution for $j = n - m$ can be shown to be

$$(19) \quad \mathcal{G}_1 = (-1)^m \cdot 2^{2n+2} \cdot n! (n + 1)! (n + m)! / (n - m)!.$$

For the integral in (18iv), we apply again repeated integrations by parts $(n + m + 1)$ -times starting with the integration of the factor $d^{n+m+1}(x^2 - 1)^{n+1}/dx^{n+m+1}$. This procedure shows that $\mathcal{G}_2 = 0$. If now the Rodrigues' formula (3) is used in (18iv) to replace the derivatives of $(x^2 - 1)^{n+1}$ and $(x^2 - 1)^n$ by appropriate Legendre functions, then vanishing of \mathcal{G}_2 can be shown to be equivalent to the following relation:

$$(20) \quad \int_{-1}^1 \frac{P_{n+1}^m(x) P_n^{m+1}(x)}{(1 - x^2)^{1/2}} dx = 0 = \mathcal{G}_2.$$

It turns out that the result in (20) is a special case of the relation (15). In view of (19) and (20), we have from (18i) and (18ii) the following result:

$$(21) \quad F_{n+1} = \int_{-1}^1 \frac{x}{1-x^2} \cdot P_n^m(x)P_{n+1}^m(x) dx = \frac{(n+m)!}{m(n-m)!}.$$

Comparing now (21) and (7ii), one finds $F_{n+1} = I_0$. Furthermore, $F_s = I_l$ for $s > n$. When these relations are combined with the result in (9), we have

$$(22) \quad F_s = I_l = I_0 = F_{n+1} = (n+m)!/m(n-m)!, \quad s > n,$$

which is the result desired in the first part [Eq. (17i)] of the theorem. The second half [Eq. (17ii)] of the theorem can be proved in a similar manner or it can be derived from (17i) by a judicious change of various indices. Thus one finds

$$(23) \quad F_s = J_l = \frac{(n+m-2l-1)!}{m(n-m-2l-1)!}, \quad s < n.$$

Similarly, in view of the relations (7i), (8i), (16), (17i) and (17ii), one may also write

$$(24i) \quad \int_{-1}^1 \frac{P_n^m(x)P_s^m(x) dx}{1-x^2} = \begin{cases} \frac{(n+m)!}{m(n-m)!} \delta_{s,n+2l}, & \text{if } s \geq n, \end{cases}$$

$$(24ii) \quad = \begin{cases} \frac{(n+m-2l)!}{m(n-m-2l)!} \delta_{s,n-2l}, & \text{if } s \leq n, \\ & l = 0, 1, 2, \dots \end{cases}$$

In [2], the above result is stated as

$$(25) \quad \int_{-1}^1 \frac{P_n^m(x)P_n^k(x)}{1-x^2} dx = \frac{(n+m)!}{m(n-m)!} \delta_{k,m},$$

which is true as it stands. However, Jones [4] in his book presents it in the following manner

$$(26) \quad \int_{-1}^1 \frac{P_n^m(x)P_s^m(x) dx}{1-x^2} = \frac{(n+m)!}{m(n-m)!} \delta_{s,n},$$

which is not correct in view of the results given here in (24i) and (24ii).

Another integral which was also used in [1] appears in Hobson's book [3] incorrectly as an exercise. This is

$$\int_{-1}^1 xP_n^m(x)P_l^m(x) dx.$$

This is not a difficult integral to evaluate. Using the last two expressions in Eq. (1) to replace $xP_n^m(x)$ and the well-known ([2], [3]) result

$$(27) \quad \int_{-1}^1 P_n^m(x)P_s^m(x) dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{s,n},$$

it can easily be shown that

$$(28) \quad \int_{-1}^1 xP_n^m(x)P_l^m(x) dx = \frac{2(l+m+1)! \delta_{n,l+1}}{(2l+1)(2l+3)(l-m)!} + \frac{2(l+m)! \delta_{n,l-1}}{(2l-1)(2l+1)(l-m-1)!}.$$

4. **Additional Results.** In this section, we present a number of results, without offering any proof since these results can be obtained from those given in Sections 2 and 3 together with some well-known relations. For $l = 0, 1, 2, \dots$, we have

$$(29) \quad \int_{-1}^1 \frac{P_n^m(x) P_{n+2l+1}^{m+1}(x) dx}{(1-x^2)^{1/2}} = \frac{2(n+m)!}{(n-m)!},$$

$$(30) \quad \int_{-1}^1 \frac{P_n^{m+1}(x) P_{n+2l+1}^m(x) dx}{(1-x^2)^{1/2}} = 0,$$

$$(31) \quad \int_{-1}^1 \frac{P_{n-1}^m(x) P_{n-2l}^{m+1}(x) dx}{(1-x^2)^{1/2}} = 0,$$

$$(32) \quad \int_{-1}^1 \frac{P_n^m(x) P_{n-2l-1}^{m-1}(x) dx}{(1-x^2)^{1/2}} = \frac{2(n+m-2l-2)!}{(n-m-2l)!},$$

$$(33) \quad \int_{-1}^1 P_{n+2l+1}^m(x) \frac{d}{dx} P_n^m(x) dx = \delta_{0,m} - \frac{(n+m)!}{(n-m)!},$$

$$(34) \quad \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n+2l+1}^m(x) dx = \delta_{0,m} + \frac{(n+m)!}{(n-m)!},$$

$$(35) \quad \int_{-1}^1 P_n^m(x) \frac{d}{dx} P_{n-2l-1}^m(x) dx = \delta_{0,m} - \frac{(n-2l-1+m)!}{(n-2l-1-m)!},$$

$$(36) \quad \int_{-1}^1 P_{n-2l-1}^m(x) \frac{d}{dx} P_n^m(x) dx = \delta_{0,m} + \frac{(n-2l-1+m)!}{(n-2l-1-m)!},$$

$$(37) \quad \int_{-1}^1 x P_n^m(x) \frac{d}{dx} P_n^m(x) dx = \delta_{0,m} - \frac{(n+m)!}{(2n+1)(n-m)!},$$

$$(38) \quad \int_{-1}^1 x P_n^m(x) \frac{d}{dx} P_{n+2l+1}^m(x) dx = 0,$$

$$(39) \quad \int_{-1}^1 x P_{n+2l+1}^m(x) \frac{d}{dx} P_n^m(x) dx = 0,$$

$$(40) \quad \int_{-1}^1 (1-x^2) P_l^m(x) \frac{d}{dx} P_n^m(x) dx = \left[\frac{2(l+1)(l+m+1)!}{(2l+1)(2l+3)(l-m)!} \delta_{n,l+1} - \frac{2(l-1)(l-m)!}{(2l-1)(2l+1)(l-m-1)!} \delta_{n,l-1} \right],$$

$$(41) \quad \int_{-1}^1 (1-x^2)^2 \frac{d}{dx} P_n^m(x) \frac{d}{dx} P_l^m(x) dx = \frac{2(l+m)!}{(2l+1)^2(l-m)!} \cdot \left\{ \frac{(l+1)^2(l+m)(l-m)}{2l-1} + \frac{l^2(l-m+1)(l+m+1)}{2l+3} \right\} \delta_{n,l} \\ - \frac{2l(l+3)(l+m+2)! \delta_{n,l+2}}{(2l+1)(2l+3)(2l+5)(l-m)!} \\ - \frac{2(l-2)(l+1)(l+m)! \delta_{n,l-2}}{(2l-3)(2l-1)(2l+1)(l-m-2)!}.$$

For $l = 1, 2, 3, \dots$, we have the following relations:

$$(42) \quad \int_{-1}^1 x P_n^m(x) \frac{d}{dx} P_{n+2l}^m(x) dx = \delta_{0,m} + \frac{(n+m)!}{(n-m)!},$$

$$(43) \quad \int_{-1}^1 x P_n^m(x) \frac{d}{dx} P_{n-2l}^m(x) dx = \delta_{0,m} - \frac{(n-2l+m)!}{(n-m-2l)!},$$

$$(44) \quad \int_{-1}^1 x P_{n+2l}^m(x) \frac{d}{dx} P_n^m(x) dx = \delta_{0,m} - \frac{(n+m)!}{(n-m)!},$$

$$(45) \quad \int_{-1}^1 x P_{n-2l}^m(x) \frac{d}{dx} P_n^m(x) dx = \delta_{0,m} + \frac{(n-2l+m)!}{(n-m-2l)!}.$$

It may be noted that one of the referees brought to the author's knowledge some related work by Power [5]. For this, the author is thankful to the referee. It appears that any related results Power presented turn out to be respective special cases of the author's. For instance, Power's equations (j), (k) and (m) are the special cases of the results presented here in (29), (6) and (24), respectively.

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